Generalized Disjunctive Programming:

Helping Creative Researchers Develop Efficient Models Systematically

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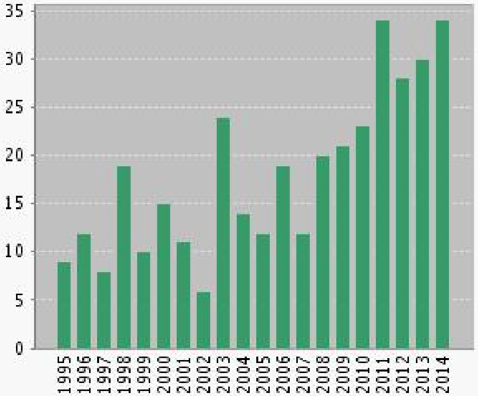
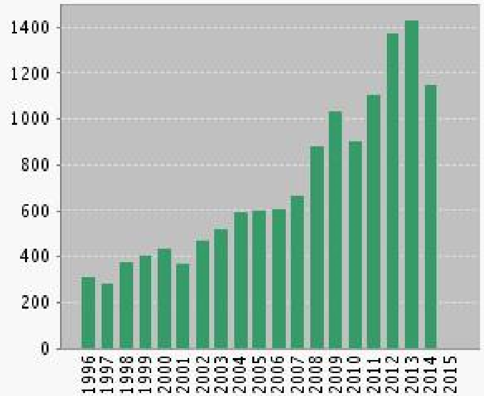
Ignacio’s citation report

Published Items in Each Year



The latest 20 years are displayed.
View a graph with all years.

Citations in Each Year



The latest 20 years are displayed.
View a graph with all years.

Results found: 460
Sum of the Times Cited [?] : 14855
Sum of Times Cited without self-citations [?] : 12531
Citing Articles [?] : 6019
Citing Articles without self-citations [?] : 5612
Average Citations per Item [?] : 32.29
h-index [?] : 64

	2011 ◀	2012	2013	2014	2015 ▶	Total	Average Citations per Year
Use the checkboxes to remove individual items from this Citation Report or restrict to items published between 1900 and 2015 Go	1108	1378	1432	1152	4	14855	412.64
1. AN OUTER-APPROXIMATION ALGORITHM FOR A CLASS OF MIXED-INTEGER NONLINEAR PROGRAMS By: DURAN, MA; GROSSMANN, IE MATHEMATICAL PROGRAMMING Volume: 36 Issue: 3 Pages: 307-339 Published: DEC 1986	17	38	28	28	0	432	14.90
2. A COMBINED PENALTY-FUNCTION AND OUTER-APPROXIMATION METHOD FOR MINLP OPTIMIZATION By: VISWANATHAN, J; GROSSMANN, IE COMPUTERS & CHEMICAL ENGINEERING Volume: 14 Issue: 7 Pages: 769-782 Published: JUL 1990	15	21	21	11	0	393	15.72
3. SIMULTANEOUS-OPTIMIZATION MODELS FOR HEAT INTEGRATION .2. HEAT-EXCHANGER NETWORK SYNTHESIS By: YEE, TF; GROSSMANN, IE COMPUTERS & CHEMICAL ENGINEERING Volume: 14 Issue: 10 Pages: 1165-1184 Published: OCT 1990	25	27	35	31	0	329	13.16
4. A STRUCTURAL OPTIMIZATION APPROACH IN PROCESS SYNTHESIS .2. HEAT-RECOVERY NETWORKS By: PAPOULIAS, SA; GROSSMANN, IE COMPUTERS & CHEMICAL ENGINEERING Volume: 7 Issue: 6 Pages: 707-721 Published: 1983	15	19	20	15	0	294	9.19
5. State-of-the-art review of optimization methods for short-term scheduling of batch processes By: Mendez, Carlos A.; Cerda, Jaime; Grossmann, Ignacio E.; et al. COMPUTERS & CHEMICAL ENGINEERING Volume: 30 Issue: 6-7 Pages: 913-946 Published: MAY 15 2006	35	44	38	31	0	281	31.22

Impact of my publications with Ignacio

- My contribution to Ignacio's h-index (64)

88.	**Two new continuous-time models for the scheduling of multistage batch plants with sequence dependent changeovers** By: Castro, Pedro M.; Grossmann, Ignacio E.; Novais, Augusto Q. INDUSTRIAL & ENGINEERING CHEMISTRY RESEARCH Volume: 45 Issue: 18 Pages: 6210-6226 Published: AUG 30 2006	13	5	4	5	0	51	5.67
91.	**New continuous-time MILP model for the short-term scheduling of multistage batch plants** By: Castro, PM; Grossmann, IE INDUSTRIAL & ENGINEERING CHEMISTRY RESEARCH Volume: 44 Issue: 24 Pages: 9175-9190 Published: NOV 23 2005	8	5	3	3	0	50	5.00

  – Zero!

Solid paper!

- Maybe we have just a few joint papers?
  – False! I am #3 in co-authors list

GROSSMANN IE (460)
CABALLERO JA (31)
MARTIN M (26)
CASTRO PM (18)
BIEGLER LT (16)
KRAVANJA Z (15)

- How about Ignacio's impact on my papers?

1.	**Simple continuous-time formulation for short-term scheduling of batch and continuous processes** By: Castro, PM; Barbosa-Povoa, AP; Matos, HA; et al. INDUSTRIAL & ENGINEERING CHEMISTRY RESEARCH Volume: 43 Issue: 1 Pages: 105-118 Published: JAN 7 2004	9	12	4	5	0	88	8.00
2.	**An improved RTN continuous-time formulation for the short-term scheduling of multipurpose batch plants** By: Castro, P; Barbosa-Povoa, APFD; Matos, H INDUSTRIAL & ENGINEERING CHEMISTRY RESEARCH Volume: 40 Issue: 9 Pages: 2059-2068 Published: MAY 2 2001	5	3	5	5	0	84	6.00
3.	**Improvements for mass-exchange networks design** By: Castro, P; Matos, H; Fernandes, MC; et al. CHEMICAL ENGINEERING SCIENCE Volume: 54 Issue: 11 Pages: 1649-1665 Published: JUN 1999	5	5	9	2	0	63	3.94

  – My top 3 papers? Before I met him!

My research work with Ignacio

- First met him in 2004
  - ESCAPE-14, 16-19 May, Portugal
- Visiting researcher to CMU
  - Sep 2004 - Jul 2005
    - Scheduling of single & multistage batch plants
      - Continuous-time, Multiple time grids, Constraint Programming, Hybrid MILP/CP
  - Ago-Dec 2008
    - Scheduling of batch & continuous plants
      - Time-dependent electricity costs, Decomposition methods, Dinkelbach algorithm
  - Oct-Nov 2011
    - Multiparametric Disaggregation Technique (lower bound)
    - Development of continuous-time models from GDP
  - Oct-Nov 2013
    - Global optimal scheduling of crude oil blending operations
      - Planning & Scheduling I, Thursday, 1:33 PM, Hilton 406-407

Heuristics for efficient MILP models

- Amongst valid choices pick exactly one (<u>exclusive OR</u>)
  - $\sum_{i,i\neq j} y_{i,j} = 1 \;\; \forall j \qquad \sum_{j,i\neq j} y_{i,j} = 1 \;\; \forall i$
    - Traveling salesman problem (one arrival & departure from each city)
- Although they look similar
  - Capacity constraints are good
    - $x_i \leq M y_i \;\; \forall i$
  - Big-M constraints are bad
    - $x_i - x_{i'} \leq M(1 - y_i) \;\forall i, i' > i$
- Adding summations makes things tighter
  - $T_{t+1} - T_t \geq \frac{\bar{\xi}_{i,t,t+1}}{\rho_i^{max}} \;\; \forall i, t \quad \Rightarrow \quad T_{t+1} - T_t \geq \sum_i \frac{\bar{\mu}_{r,i}\bar{\xi}_{i,t,t+1}}{\rho_i^{max}} \;\; \forall r, t$
    - Timing constraints by equipment unit r rather than task i (Castro et al. 2004)
  - Excess resource balances (Pantelides, 1994)

	RMIP	Solution	CPUs
Before	7273	2601	52.2
After	2689		15.6
Before	7361	2638	200,652
After	2695		1913

Generalized Disjunctive Programming:
Systematic Modeling Framework to Derive MILPs

Disjunctive Programming (Balas, 1979)

- Most natural & straightforward way of stating problems involving logical conditions
- Any mixed-integer program can be stated as a DP, usually in more than one way
- Various formulations may give rise to linear relaxations of varying strengths
  - Affects computational performance

At least one subset of constraints $h_{i,k}(x) \leq b_{i,k}$ must hold

$$\min f(x)$$
$$s.t.\ g(x) \leq 0$$
$$\boxed{\bigvee_{i \in D_k} \left[A_{i,k} x \leq b_{i,k} \right]} \quad \forall \boxed{k \in K}$$
$$x \in \mathbb{R}^n, x \geq 0$$

(DP)

Terms of disjunction k

Set of disjunctions

- No 0-1 variables explicitly included in the model

Generalized Disjunctive Programming

- Logic-based modelling framework (Raman & Grossmann, 1994)
  - 220 citations, #8 (web of Science)
- Adds Boolean variables and logic propositions
  - Selection of a given term in a disjunction may affect other constraints
    - Chemical process synthesis (Raman & Grossmann, 1991)

Boolean variables appear explicitly inside disjunctive terms

$$
\begin{aligned}
&\min f(x) \\
&s.t.\ g(x) \le 0 \\
&\bigvee_{i \in D_k} \begin{bmatrix} \boxed{Y_{i,k}} \\ A_{i,k} x \le b_{i,k} \end{bmatrix} \quad \forall k \in K \\
&\boxed{\Omega(Y) = True} \\
&x \in \mathbb{R}^n, x \ge 0, \boxed{Y \in \{False, True\}^m}
\end{aligned}
\qquad \textbf{(GDP)}
$$

Logic propositions

Direct correspondence with $\{0,1\}$ variables y

Reformulation of a GDP into MILP

- Disjunctions (Balas, 1985)
  - Big-M
    - Simplest form but yields poor relaxations

$$A_{i,k}x \le b_{i,k} + M_{i,k}(1 - y_{i,k}) \ \ \forall k \in K, i \in D_k$$

$$\boxed{M_{i,k} = \max\{A_{i,k}x - b_{i,k}: 0 \le x^L \le x \le x^U\} \ \ \forall k \in K, i \in D_k}$$

Tightest big-M parameters

  - Convex hull
    - At least as tight as big-M but increases problem size and is much harder

$$A_{i,k}\hat{x}_{i,k} \le b_{i,k}y_{i,k} \ \ \forall k \in K, i \in D_k$$

$$x = \sum_{i \in D_k} \boxed{\hat{x}_{i,k}} \ \forall k \in K$$

Tighter bounds

$$\boxed{\hat{x}^L_{i,k}} y_{i,k} \le A_{i,k}\hat{x}_{i,k} \le \boxed{\hat{x}^U_{i,k}} y_{i,k} \ \ \forall k \in K, i \in D_k$$

Disaggregated variables (new set)

  - Common constraint

$$\sum_{i \in D_k} y_{i,k} \ge 1 \ \ \forall k \in K$$

- Logic propositions
  - Replace with linear inequalities (Clocksin & Mellish, 1981)

Remarks about big-M way of modeling

- Used extensively by many researchers
  - Avoids bilinear terms

$$IF\ y_{i,k} = 1\ THEN\ A_{i,k}x \le b_{i,k}$$

Worst

$$A_{i,k}\boxed{x y_{i,k}} \le b_{i,k} y_{i,k}$$

Holds true for $y_{i,k} = 0$ regardless of x

Better

$$A_{i,k}x \le b_{i,k} + M_{i,k}(1 - y_{i,k})$$

- Few bother to calculate $M_{i,k}$ (global value M used instead)
  - Output from big-M reformulation might not be a big-M constraint
    - Example from scheduling with multiple time grids (Castro & Grossmann, 2012)

$$\bigvee_i \begin{bmatrix} y_{i,k} \\ x_{k+1} - x_k \ge p_i \\ x_k \ge r_i \end{bmatrix} \forall k \Rightarrow \begin{array}{r} x_{k+1} - x_k \ge p_i - M_{i,k}^1(1 - y_{i,k})\ \forall i,k \\ x_k \ge r_i - M_{i,k}^2(1 - y_{i,k})\ \forall i,k \end{array} \Rightarrow \begin{array}{r} x_{k+1} - x_k \ge p_i \cdot y_{i,k} \forall i,k \\ x_k \ge r_i \cdot y_{i,k}\ \forall i,k \end{array}$$

But we can infer that

$$M_{i,k}^1 = \max\{x_k - x_{k+1} + p_i : x_k - x_{k+1} \le 0\} = p_i$$

$$M_{i,k}^2 = \max\{-x_k + r_i : x_k \ge 0\} = r_i$$

Remarks about convex hull reformulation

- Always check if disaggregated variables can be removed
  - Compact or sharp formulation (Jeroslow and Lowe, 1984)
    - Example 1: Modeling a semi-continuous variable
      - Flow in a unit only if the unit is selected from the superstructure

$$\begin{bmatrix} y \\ x^L \le x \le x^U \end{bmatrix} \bigvee \begin{bmatrix} \neg y \\ x = 0 \end{bmatrix} \Rightarrow \begin{array}{l} x = \hat{x}^1 + \hat{x}^2 \\ x^L y \le \hat{x}^1 \le x^U y \\ \hat{x}^2 = 0 \cdot (1-y) \end{array} \Rightarrow x^L y \le x \le x^U y$$

    - Example 2: Scheduling with multiple time grids revisited

$$\bigvee_i \begin{bmatrix} y_{i,k} \\ x_{k+1} - x_k \ge p_i \\ x_k \ge r_i \end{bmatrix} \forall k \Rightarrow \begin{array}{l} x_k = \sum_i \hat{x}_{k,i,k} \quad \forall k \\ x_{k+1} = \sum_i \hat{x}_{k+1,i,k} \quad \forall k \\ \hat{x}_{k+1,i,k} - \hat{x}_{k,i,k} \ge p_i \cdot y_{i,k} \quad \forall i,k \\ \hat{x}_{k,i,k} \ge r_i \cdot y_{i,k} \quad \forall i,k \end{array} \Rightarrow \begin{array}{r} x_{k+1} - x_k \ge \boxed{\sum_i p_i \cdot y_{i,k} \ \forall k} \\ x_k \ge \boxed{\sum_i r_i \cdot y_{i,k} \ \forall k} \end{array}$$

Similar to big-M

$$\boxed{\sum_i y_{i,k} = 1 \quad \forall k} \rightarrow$$

Shared by convex hull & compact convex hull

CPUs	Big-M (up to 1 h)	Convex hull	Compact Convex hull
EX6	Suboptimal	123	1.53
EX7	No solution	21.6	1.07
EX8	No solution	80.3	1.80
EX9	No solution	63.3	0.59
EX10	No solution	213	3.91

Advantages of modeling with GDP

- GDP model much easier to understand
  - Constraints
    - In their simplest (linear) form
    - Fewer in number
  - Particularly relevant for complex problems
    - Not uncommon to find research papers featuring MILP models with 50-100 constraints, mostly big-M
      - How can one assess model efficiency?
- Conversion to MILP format can be automated
  - LogMIP for GAMS (Vecchietti & Grossmann, 2007)
    - Works for very simple cases, cannot handle disjunctions over a set
- Useful guidelines for hands-on approach needed
  - When is it worth to derive convex hull reformulation?

Worth to derive convex hull?

- General precedence

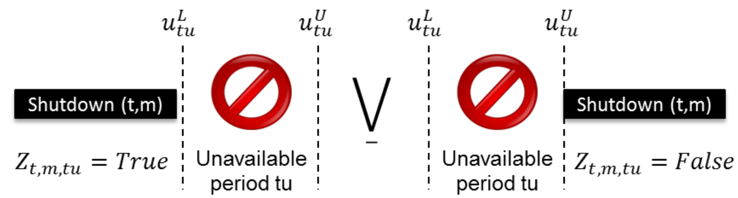
$$\begin{bmatrix} Y_{t,m,t',m'} \\ \boxed{Tm_{t,m}} + sd_t \le \boxed{Tm_{t',m'}} \end{bmatrix} \underline{\bigvee} \begin{bmatrix} \neg Y_{t,m,t',m'} \\ Tm_{\boxed{t',m'}} + sd_{t'} \boxed{\le} Tm_{\boxed{t,m}} \end{bmatrix} \quad \forall\, t, t', m' > m$$

Two variables involved   Similar constraint but the indices have changed

  - Answer: No (disaggregated variables with 6 indices)

- Forbid task execution on a given time window

$$\begin{bmatrix} Z_{t,m,tu} \\ \boxed{Tm_{t,m}} + sd_t \le u_{tu}^L \end{bmatrix} \underline{\bigvee} \begin{bmatrix} \neg Z_{t,m,tu} \\ Tm_{t,m} \boxed{\ge} u_{tu}^U \end{bmatrix} \quad \forall\, t, m, tu$$



Single variable   Different constraint

  - Maybe (disaggregated variables with 3 indices)

- Single task per unit

$$\underline{\bigvee}_{\boxed{i}} \begin{bmatrix} y_{i,k} \\ \boxed{x_{k+1} - x_k} \ge p_i \\ x_k \ge r_i \end{bmatrix} \forall k$$

Variables index different than disjunction index

  - Absolutely, no additional variables and constraints

Success Cases:
Optimization Approaches

Relaxation of bilinear term $z_{ij} = x_i x_j$

- Tightest continuous relaxation
  - McCormick (1976)

Underestimators

$$z_{ij} \geq x_i \cdot x_j^L + x_i^L \cdot x_j - x_i^L \cdot x_j^L$$

$$z_{ij} \geq x_i \cdot x_j^U + x_i^U \cdot x_j - x_i^U \cdot x_j^U$$

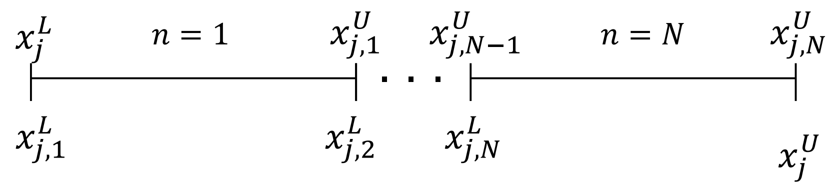
$$z_{ij} \leq x_i \cdot x_j^L + x_i^U \cdot x_j - x_i^U \cdot x_j^L$$

$$z_{ij} \leq x_i \cdot x_j^U + x_i^L \cdot x_j - x_i^L \cdot x_j^U$$

Overestimators

- Piecewise McCormick envelopes
  - Bergamini et al. (2005)

Domain of x_j divided into N partitions



Single active partition $\quad y_{j,1} = true \;\underline{\vee} \cdots \underline{\vee}\; y_{j,N} = true$

Partition dependent bounds for x_j

Disjunction index (n) $\neq$ variables index (i,j)

$$\boxed{\bigvee_n} \begin{bmatrix} y_{j,n} \\ z_{ij} \geq x_i \cdot \boxed{x_{j,n}^L} + x_i^L \cdot x_j - x_i^L \cdot \boxed{x_{j,n}^L} \\ z_{ij} \geq x_i \cdot \boxed{x_{j,n}^U} + x_i^U \cdot x_j - x_i^U \cdot \boxed{x_{j,n}^U} \\ z_{ij} \leq x_i \cdot \boxed{x_{j,n}^L} + x_i^U \cdot x_j - x_i^U \cdot \boxed{x_{j,n}^L} \\ \boxed{z_{ij}} \leq \boxed{x_i} \cdot \boxed{x_{j,n}^U} + x_i^L \cdot \boxed{x_j} - x_i^L \cdot \boxed{x_{j,n}^U} \\ x_{j,n}^L \leq x_j \leq x_{j,n}^U \end{bmatrix} \quad \forall i,j$$

$$x_i^L \leq x_i \leq x_i^U$$

$$x_{j,n}^L = x_j^L + (x_j^U - x_j^L) \cdot (n-1)/N$$

$$x_{j,n}^U = x_j^L + (x_j^U - x_j^L) \cdot n/N$$

PCM relaxation of bilinear program

- MILP is derived from convex hull reformulation

$$\min z^R = f_0(x) = \sum_{(i,j)\in BL} a_{ij0} w_{ij} + h_0(x)$$

$$f_q(x) = \sum_{(i,j)\in BL} a_{ijq} w_{ij} + h_q(x) \le 0 \ \ \forall q \in Q\backslash\{0\}$$

$$\left.\begin{aligned}
z_{ij} &\ge \boxed{\sum_{n=1}^{N} \left(\hat{x}_{ijn}\cdot x_{jn}^L + x_i^L \cdot \hat{x}_{jn} - x_{ijn}^L \cdot x_{jn}^L \cdot y_{jn}\right)} \\
z_{ij} &\ge \sum_{n=1}^{N} \left(\hat{x}_{ijn}\cdot x_{jn}^U + x_i^U \cdot \hat{x}_{jn} - x_{ijn}^U \cdot x_{jn}^U \cdot y_{jn}\right) \\
z_{ij} &\le \sum_{n=1}^{N} \left(\hat{x}_{ijn}\cdot x_{jn}^L + x_i^U \cdot \hat{x}_{jn} - x_{ijn}^U \cdot x_{jn}^L \cdot y_{jn}\right) \\
z_{ij} &\le \sum_{n=1}^{N} \left(\hat{x}_{ijn}\cdot x_{jn}^U + x_i^L \cdot \hat{x}_{jn} - x_{ijn}^L \cdot x_{jn}^U \cdot y_{jn}\right) \\
x_i &= \sum_{n=1}^{N} \hat{x}_{ijn}
\end{aligned}\right\} \forall\, (i,j) \in BL$$

Summation eliminates the need for disaggregated variables linked to z_{ij}

$$\left.\begin{aligned}
x_{jn}^L &= x_j^L + \frac{\left(x_j^U - x_j^L\right)\cdot(n-1)}{N} \\
x_{jn}^U &= x_j^L + \frac{\left(x_j^U - x_j^L\right)\cdot n}{N} \\
x_{jn}^L \cdot y_{jn} &\le \hat{x}_{jn} \le x_{jn}^U \cdot y_{jn}
\end{aligned}\right\} \forall\, \{j|(i,j) \in BL\}, n \in \{1, \dots, N\}$$

$$x_i^L \cdot y_{jn} \le \hat{x}_{ijn} \le x_i^U \cdot y_{jn} \ \forall\, (i,j) \in BL, n \in \{1, \dots, N\}$$

$$\left.\begin{aligned}
x_j &= \sum_{n=1}^{N} \hat{x}_{jn} \\
\sum_{n=1}^{N} y_{jn} &= 1
\end{aligned}\right\} \forall\, \{j|(i,j) \in BL\}$$

$$x^L \le x \le x^U$$
$$x \in \mathbb{R}^m$$
$$y_{jn} \in \{0,1\} \ \forall\, \{j|(i,j) \in BL\}, n \in \{1, \dots, N\}$$

Bilinear term approximation $\tilde{z}_{ij} \approx z_{ij} = x_i x_j$

- Multiparametric disaggregation (Teles et al. 2013)

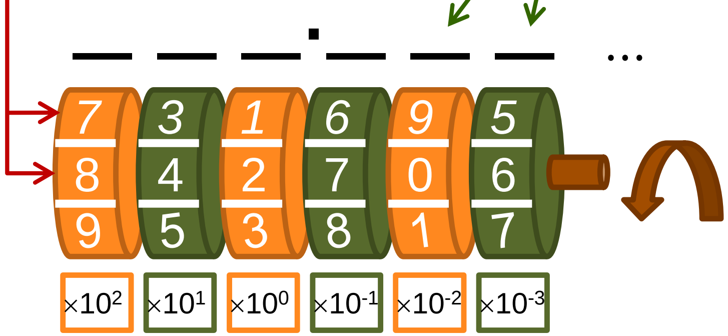
$$\tilde{z}_{ij} = x_i \tilde{x}_j = x_i \sum_{k=\boxed{\psi}}^{\boxed{\eta}} \tilde{x}_{j,k}$$

$$\eta = \lfloor \log_{10} x_j^U \rfloor$$

Defines accuracy level

MDT discretizes domain of variable x_j up to a certain accuracy

$$\bigvee_{p=0}^{9} \left[\begin{array}{c} y_{j,p,k} \\ \tilde{x}_{j,k} = \boxed{p} \cdot 10^k \\ \tilde{z}_{ij,k} = x_i \cdot p \cdot 10^k \end{array} \right] \boxed{\forall k}$$



Convex hull

Complete MILP formulation

$$\tilde{x}_j = \sum_k \sum_p p \cdot 10^k \cdot y_{j,p,k}$$

$$\tilde{z}_{ij} = \sum_k \sum_p \hat{x}_{i,p,k} \cdot p \cdot 10^k$$

$$x_i = \sum_p \hat{x}_{i,p,k} \quad ; \quad \sum_p y_{j,p,k} = 1 \quad , \forall k$$

$$x_i^L \cdot y_{j,p,k} \leq \hat{x}_{i,p,k} \leq x_i^U \cdot y_{j,p,k} \text{ , } \forall p, k$$

$$x_j^L \leq \tilde{x}_j \leq x_j^U$$

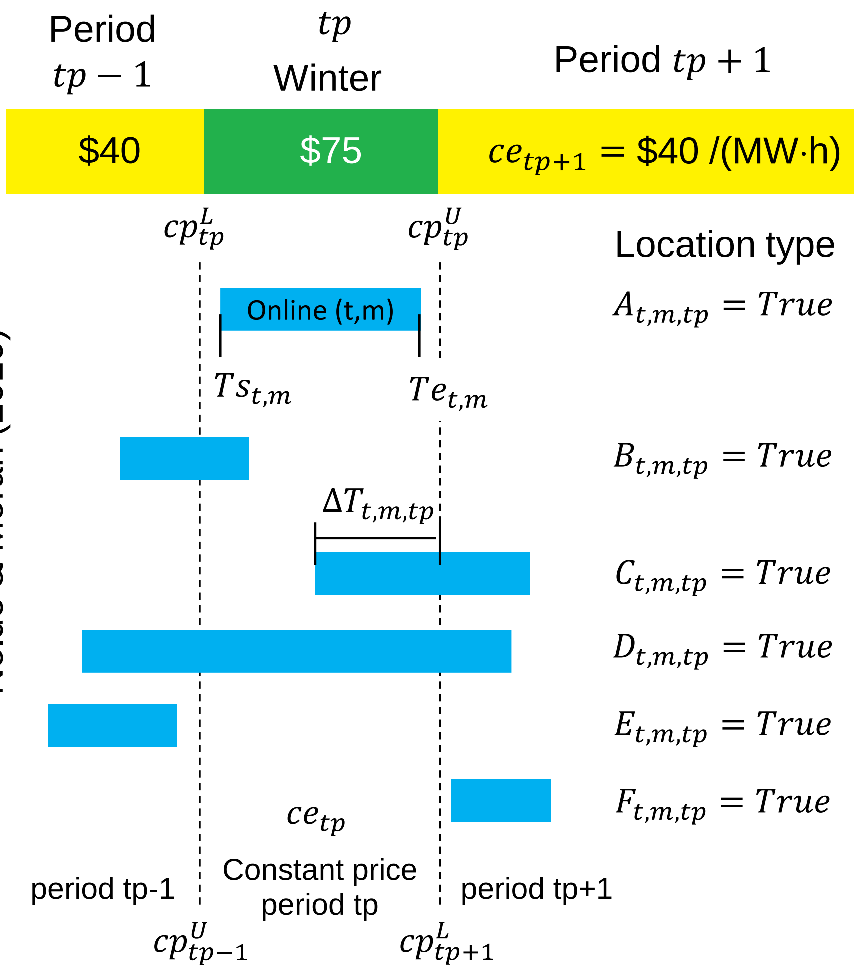
$$\hat{x}_{i,p,k} \geq 0 \text{ ; } y_{j,p,k} \in \{0,1\}, \forall p, k$$

Success cases:

Industrial scheduling problems dealing with energy

Seasonal electricity tariffs in a power plant

Nolde & Morari (2010)



$$\begin{bmatrix} A_{t,m,tp} \\ Ts_{t,m} \geq cp_{tp}^{L} \\ Ts_{t,m} \leq cp_{tp}^{U} \\ Te_{t,m} \geq cp_{tp}^{L} \\ Te_{t,m} \leq cp_{tp}^{U} \\ \Delta T_{t,m,tp} = P_{t,m} \end{bmatrix} \veebar \begin{bmatrix} B_{t,m,tp} \\ Ts_{t,m} \leq cp_{tp}^{L} \\ (Ts_{t,m} \leq cp_{tp}^{U}) \\ Te_{t,m} \geq cp_{tp}^{L} \\ Te_{t,m} \leq cp_{tp}^{U} \\ \Delta T_{t,m,tp} = Te_{t,m} - cp_{tp}^{L} \end{bmatrix} \veebar$$

$$\veebar \begin{bmatrix} C_{t,m,tp} \\ Ts_{t,m} \geq cp_{tp}^{L} \\ Ts_{t,m} \leq cp_{tp}^{U} \\ (Te_{t,m} \geq cp_{tp}^{L}) \\ Te_{t,m} \geq cp_{tp}^{U} \\ \Delta T_{t,m,tp} = cp_{tp}^{U} - Ts_{t,m} \end{bmatrix} \veebar \begin{bmatrix} D_{t,m,tp} \\ Ts_{t,m} \leq cp_{tp}^{L} \\ (Ts_{t,m} \leq cp_{tp}^{U}) \\ (Te_{t,m} \geq cp_{tp}^{L}) \\ Te_{t,m} \geq cp_{tp}^{U} \\ \Delta T_{t,m,tp} = cp_{tp}^{U} - cp_{tp}^{L} \end{bmatrix} \veebar$$

$$\veebar \begin{bmatrix} E_{t,m,tp} \\ (Ts_{t,m} \leq cp_{tp}^{L}) \\ (Ts_{t,m} \leq cp_{tp}^{U}) \\ Te_{t,m} \leq cp_{tp}^{L} \\ (Te_{t,m} \leq cp_{tp}^{U}) \\ \Delta T_{t,m,tp} = 0 \end{bmatrix} \veebar \begin{bmatrix} F_{t,m,tp} \\ (Ts_{t,m} \geq cp_{tp}^{L}) \\ Ts_{t,m} \geq cp_{tp}^{U} \\ (Te_{t,m} \geq cp_{tp}^{L}) \\ (Te_{t,m} \geq cp_{tp}^{U}) \\ \Delta T_{t,m,tp} = 0 \end{bmatrix} \quad \forall\, t, m, tp$$

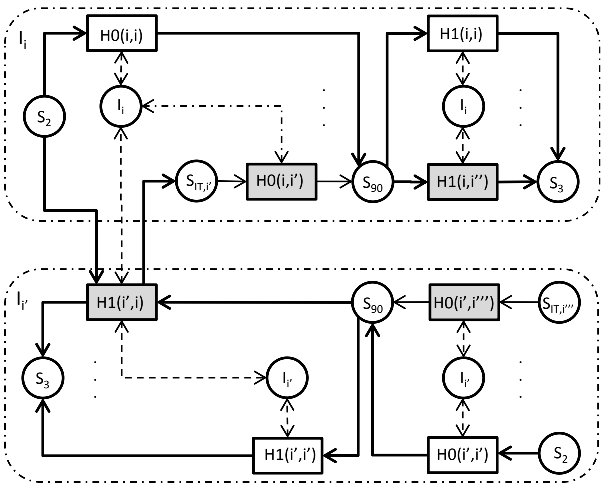
Tightening constraint $\sum_{tp} \Delta T_{t,m,tp} \leq P_{t,m} \ \forall\, t, m$

$$A_{t,m,tp} + B_{t,m,tp} + C_{t,m,tp} + D_{t,m,tp} + E_{t,m,tp} + F_{t,m,tp} = 1$$

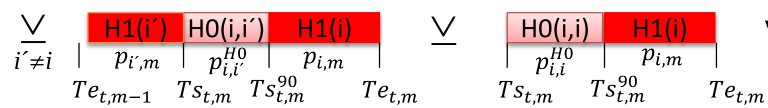
$$\max Revenue = \sum_{t} \sum_{m} \sum_{tp} \Delta T_{t,m,tp} \cdot ce_{tp} \cdot pw_{m}$$

Steam sharing in a pulp plant

- Multipurpose plant, single grid, RTN-based approach
  - Superstructure (2001)



- Multistage plant, multiple time grids, GDP-based
  - Embedded disjunctions (2013)
    - Wednesday, 5 PM, Hilton 406-7



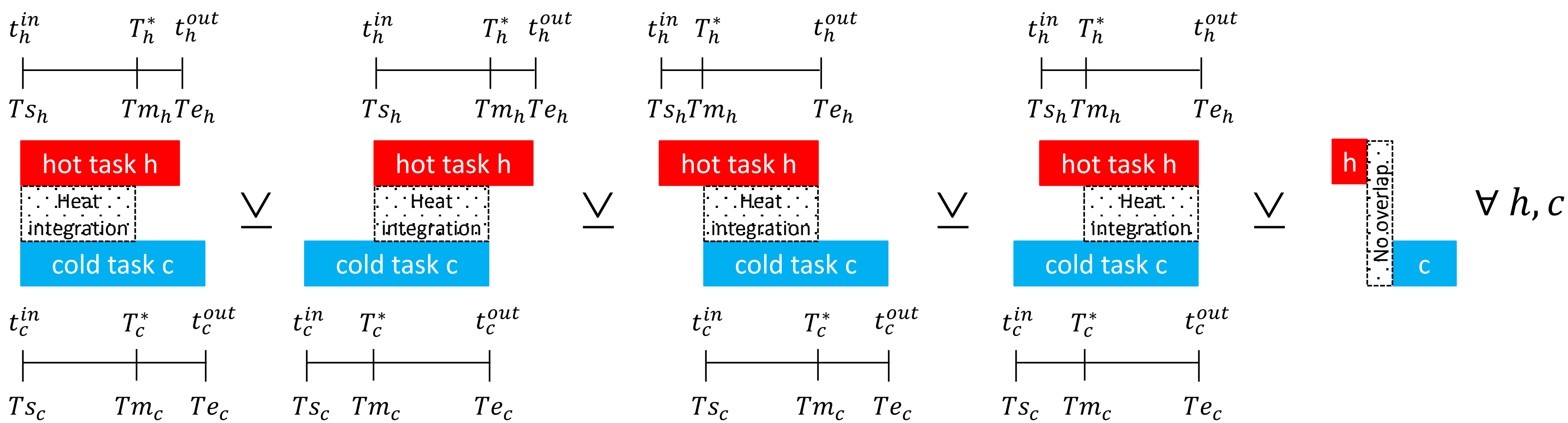
$$\underset{i'\neq i}{\veebar} \quad Z_{i,i',t} = True \qquad \veebar \qquad Z_{i,i,t} = True \qquad \forall i \in I,\ m \in M^{SL},\ t \in T$$

$$\bigvee_{i} \left[\begin{array}{c} Y_{i,t} \\ Te_{t,m} = Ts^{90}_{t,m} + p_{i,m} \\ \underset{i' \neq i}{\veebar} \begin{bmatrix} Z_{i,i',t} \\ Ts^{90}_{t,m} = Ts_{t,m} + p^{H0}_{i,i'} \\ Te_{t,m-1} \le Ts_{t,m} - p_{i',m} \end{bmatrix} \veebar \begin{bmatrix} Z_{i,i,t} \\ Ts^{90}_{t,m} = Ts_{t,m} + p^{H0}_{i,i} \end{bmatrix} \end{array} \right] \forall m,t$$

Approach	Discrete-time	Continuous-time		Discrete-time	Continuous-time	
		Single grid	Multiple grids		Single grid	Multiple grids
Binary variables	33263	2688	79	31583	3584	79
Constraints	30932	3331	132	29372	4435	132
Cycle time H (min)	**594**	606	**594**	**564**	571	**564**
Total CPU (s)	563	576	**0.11**	217	25252	**0.21**

Direct heat integration in a batch plant

- Extension of Yee et al. (1990) model for continuous plants
  - Two-stage heating/cooling with matches in parallel
  - Possible interactions between hot & cold stream modelled with GDP



$Y_{h,c}^{ss} = True$ $\quad$ $Y_{h,c}^{se} = True$ $\quad$ $Y_{h,c}^{es} = True$ $\quad$ $Y_{h,c}^{ee} = True$ $\quad$ $Y_{h,c}^{no} = True$

$$\begin{bmatrix} Y_{h,c}^{ss} \\ Ts_h = Ts_c \\ Tm_h = Tm_c \\ T_h^* \geq T_c^* + \Delta t \\ Q_{h,c}^{ss} \leq q_{h,c} \\ Q_{h,c}^{se}, Q_{h,c}^{es}, Q_{h,c}^{ee} = 0 \end{bmatrix} \bigvee \begin{bmatrix} Y_{h,c}^{se} \\ Ts_h = Tm_c \\ Tm_h = Te_c \\ T_h^* \geq t_c^{out} + \Delta t \\ Q_{h,c}^{se} \leq q_{h,c} \\ Q_{h,c}^{ss}, Q_{h,c}^{es}, Q_{h,c}^{ee} = 0 \end{bmatrix} \bigvee \begin{bmatrix} Y_{h,c}^{es} \\ Tm_h = Ts_c \\ Te_h = Tm_c \\ t_h^{out} \geq T_c^* + \Delta t \\ Q_{h,c}^{es} \leq q_{h,c} \\ Q_{h,c}^{ss}, Q_{h,c}^{se}, Q_{h,c}^{ee} = 0 \end{bmatrix} \bigvee \begin{bmatrix} Y_{h,c}^{ee} \\ Tm_h = Tm_c \\ Te_h = Te_c \\ t_h^{out} \geq t_c^{out} + \Delta t \\ Q_{h,c}^{ee} \leq q_{h,c} \\ Q_{h,c}^{ss}, Q_{h,c}^{se}, Q_{h,c}^{es} = 0 \end{bmatrix} \bigvee \begin{bmatrix} Y_{h,c}^{no} \\ Ts_h \geq 0 \\ Ts_c \geq 0 \\ Q_{h,c}^{ss}, Q_{h,c}^{se}, Q_{h,c}^{es}, Q_{h,c}^{ee} = 0 \end{bmatrix} \quad \forall\, h, c$$

Conclusions

- Overview of the basics of linear GDP
- Identification of desired type of disjunctions
  - Leading to sharp (compact) convex hull reformulations
- Examples of global optimization approaches that validate such analysis
  - Piecewise McCormick envelopes
  - Multiparametric Disaggregation Technique
- GDP makes it easier to model complex scheduling problems
  - Alternative formulation with improved performance
    - Orders of magnitude

- Much more about GDP
  - Recent reviews by Grossmann & Trespalacios (2013,14)
    - Further tightening through basic steps
    - Logic based algorithms for convex GDP
    - Non-convex GDP
- Several research groups worldwide rely on GDP for modeling
  - Many more need to realize how powerful it is!